Fn isomorphic to A(V)

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Fn isomorphic to A(V)

Theorem:The set Fn of all nxn matrices over F is an algebra over F. If V is an n-dimensional vector space over F then A(V) and Fn are isomorphic as algebras over F

Proof: Given: dim (V) = n where V is vector space over F

$\Rightarrow$ dim [A(V)] = n^2

Let $T \in A(V)$

Let $\{v_1, v_2, \text{-----} v_n\}$ be a fixed basis of V

Now, v_iT is uniquely expressible as a linear combination of the basis elements $v_1, v_2, \ldots, v_n$

$$\Rightarrow \boxed{v_i T = \sum_{j=1}^{n} a_{ij} v_j} \quad (i = 1, \ldots n) \rightarrow \text{(i)}$$

So each T $\in$ A(V) has associated with it a unique matrix

$$m(T) = \begin{pmatrix} a_{11} & a_{12} & \ldots a_{1n} \\ a_{21} & a_{22} & \ldots a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} & \ldots a_{nn} \end{pmatrix} = (a_{ij})_{nxn} \text{ over F.}$$

This m(T) is called the matrix of the linear Transformation T $\in$ A(V) relative to the basis $\{v_1,\ldots,v_n\}$ of V

Conversely if A = (a_{ij}) is a given nxn matrix over F. Then, for a given basis $\{v_1,\ldots,v_n\}$ of V, if we define $T:V \to V$ by $v_iT = \sum_{j=1}^{n} a_{ij} v_j$, (i =1 to n), then T becomes a Linear Transformation on V.

Let, $F_n = \{(a_{ij})_{nxn} / a_{ij} \in F\}$

Let $(a_{ij}), (b_{ij}) \in F_n$.

Then $(a_{ij}) = (b_{ij})$ iff $a_{ij} = b_{ij}\ \forall\ i,j$

Now, consider the mapping

$A(V) \to F_n$ defined by $\boxed{T \to m(T) = (a_{ij})} \to (2)$

This is a one – one mapping of A(V) onto F_n.

$\Rightarrow$ we can define +,multiplication, scalar multiplication on F_n, since A(V) is an algebra.

(i) Addition in F_n

Let $A = (a_{ij})$ & $B = (b_{ij})$ be element in F_n. Suppose, further that, under the mapping (2),

$T \mapsto A$ & $S \mapsto B$.

Then, $v_i T = \sum_{j=1}^{n} a_{ij} v_j$ &

$$v_i S = \sum_{j=1}^{n} b_{ij} v_j$$

so that $A = m(T)$ & $B = m(S)$

Now, by the definition of addition of Linear transformation it follows that,

$v_i (T+S) = v_i T + v_i S$

$$= \sum a_{ij} v_j + \sum b_{ij} v_j$$

$\therefore$ we see that under the mapping (2)

$T + S \mapsto (a_{ij} + b_{ij})$,

we define addition in F_n as follows:

$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \longrightarrow (3) \qquad \Rightarrow \; m(T) + m(S) = m(T+S)$

(ii) <u>Multiplication in F_n</u>

By the definition of a product of linear transformation in A(V), we have.

$$v_i (TS) = (v_i T)S$$

$$= (\sum_k a_{ik} v_k) S$$

$$= \sum_k a_{ik} (v_k S)$$

$$= \sum_k a_{ik} (\sum_j b_{kj} v_j)$$

$$\Rightarrow v_i (TS) = \sum_j (\sum_k a_{ik} b_{kj}) v_j$$ [By rearranging the order of summation]

Hence under the mapping (2),

$$\therefore TS \mapsto (\sum_k a_{ik} b_{kj})$$

Accordingly we definition multiplication in F_n as follows:

$$(a_{ij}) (b_{ij}) = (\sum_k a_{ik} b_{kj}) \text{ ------ } (4)$$

i.e., m(T) . m(S) = m(TS)

(iii) Scalar multiplication in F_n

If $c \in F$, we have by the definition of scalar multiplication in A(V),

$$v_i (cT) = c(v_i T)$$

$$= c[\sum_j a_{ij} v_j]$$

$$= \sum_j (ca_{ij}) v_j$$

Accordingly we define scalar multiplication in F_n as follows:

$c(a_{ij}) = (ca_{ij})$ -------- (5)

ie) $m(cT) = cm(T)$

We have now defined addition multiplication and scalar multiplication in F_n in such a way that, all of these operations are preserved under the mapping (2)

That is, if under this mapping $T \mapsto m(T)$ & $S \mapsto m(S)$

$$\text{then,} \quad T+S \mapsto m(T+S) = m(T) + m(S)$$
$$TS \mapsto m(TS) = m(T)\, m(S)$$
$$cT \mapsto m(cT) = c\,m(T) \qquad \text{for } C \in F$$

Thus we have shown that the mapping (equation (2) is an isomorphism of A (V) onto F_n as algebras)

Hence the set F_n of all nxn matrices over F is an algebra over F. If V is an n – dimensional vector space over F, then A(V) and F_n are isomorphic as algebras over F. ②

Definition:

(i) Zero Matrix:

* Zero matrix is a matrix all of whose entries are zero.
* The Zero element of an algebra F_n is the nxn zero matrix.

(ii) Unit matrix:

- Unit matrix is the matrix whose diagonal elements are one and whose entries elsewhere are zero.
- We write it as 'I'.
- The unit element of F_n under multiplication is I.

(iii) Scalar matrix:

If $c \in F$ then, cI is called Scalar matrix.

Example:

$$cI = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}$$

(iv) <u>Triangular matrix:</u>

The matrix $A \in F_n$ is called triangular if all the entries above the main diagonal are zero (0).

If all the entries below the main diagonal are zero, the matrix is also called Triangular.

(v) <u>Invertible (or) Regular (or) non – Singular Matrix</u>:

The matrix $A \in F_n$ is called invertible or non – singular if there exists $B \in F_n$ such that $AB = BA = I$ the matrix B is called the inverse of A and $A^{-1} = B$

<u>Note:</u>

Since $A(V) \approx F_n$, $T \in A(V)$ is invertible iff $m(T)$ has inverse in F_n.

Theorem:

Let V be a vector space of dimension n over F and let $T \in A(V)$. If $m_1(T)$ and $m_2(T)$ are the matrices of T, relative to two bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ of V, respectively. There is an invertible matrix C in F_n such that $m_2(T) = Cm_1(T)C^{-1}$.

Proof:

Given, dim (V) =n; $T \in A(V)$;

$m_1(T)$ is the matrix corresponding to $\{v_1, \ldots, v_n\}$

$m_2(T)$ is the matrix corresponding to $\{w_1, \ldots, w_n\}$

Let $m_1(T) = (a_{ij})$ where $v_iT = \sum_{j=1}^{n} a_{ij} v_j$

Let $m_2(T) = (b_{ij})$ where $w_iT = \sum_{j=1}^{n} b_{ij} w_j$

We define

$S : V \to V$ by $v_iS = w_i$ $(i = 1 \text{ to } n)$ then S is a Linear Transformation on V.

Claim: S is onto

Let $y \in V$ (co domain)

$\Rightarrow y = c_1 w_1 + c_2 w_2 + \text{-----} + c_n w_n$.

Let $x = c_1 v_1 + c_2 v_2 + \text{-----} + c_n v_n$

Then $x \in V$

Now $xS = (c_1 v_1 + \text{-----} + c_n v_n) S$.

$= c_1 (v_1 S) + \text{-----} + c_n (v_n S)$

$= c_1 w_1 + c_2 w_2 + \text{-----} + c_n w_n = y$

.: for all $y \in V$ there exists $x \in V$ such that, $xS = y$

.: S is onto $\Rightarrow$ S is invertible

$\Rightarrow S^{-1}$ exists

Now, $w_i T = \sum_{j=1}^{n} b_{ij} w_j$

$$(v_i S) T = \sum_{j=1}^{n} b_{ij} (v_j S) \quad \text{(Since } w_i = v_i S\text{)}$$

$$\Rightarrow v_i (ST) = \sum_{j=1}^{n} (b_{ij} v_j) S \quad [\because S \text{ is linear }]$$

$$v_i (STS^{-1}) = \sum_{j=1}^{n} b_{ij} v_j \quad (\because SS^{-1} = I)$$

$$\Rightarrow m_1 (STS^{-1}) = (b_{ij}) = m_2 (T)$$

We know that

$T \mapsto m_1(T)$ is an isomorphism of $A(V)$ onto F_n. i.e., $A(V) \approx F_n$.

$\therefore\ m_1(S)\, m_1(T)\, m_1(S^{-1}) = m_2(T)$

$\Rightarrow \quad C\, m_1(T)\, C^{-1} = m_2(T)$

Where $C = m_1(S) \in F_n$ is invertible.

Hence the theorem is proved. The matrix $C = m_1(S)$ is called the matrix of the change of basis. ②

TRIANGULAR FORMS.

Definition:

Invariant subspace.

Let V be n – dimensional vector space. over F. Let W be a subspace of V & let $T \in A(V)$ then W is invariant under T if $WT \subseteq W$

Lemma:

If the subspace W of V is invariant under $T \in A(V)$, then T induces a linear transformation $\bar{T}$ on the quotient space V/W defined by $(v+W)\bar{T} = vT + W$

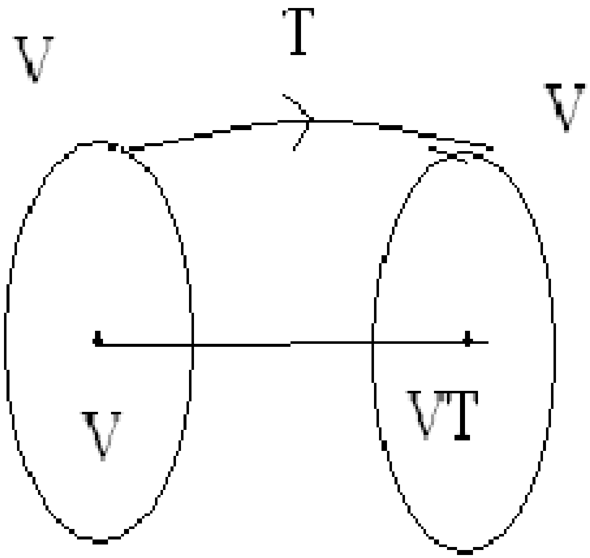
If $p_1(x)$ is the minimal polynomial of $\bar{T}$ over F, and if $p(x)$ is that for T then, $p_1(x) \mid p(x)$.

Proof

Given: W is an invariant subspace of V and $T \in A(V)$

$\Rightarrow$ W T $\subseteq$ W

We know that $V/W = \{v+W \,/\, v \in V\}$



We know that $T \in A(V) \Rightarrow T:V \rightarrow V$ is Linear Transformation.

$\Rightarrow vT \in V$

$\Rightarrow vT + W \in V/W$

Define $\overline{T} : V/W \rightarrow V/W$ by $(v+W)\,\overline{T} = vT + W \;\; \forall \;\; v+W \in V/W, v \in V$

Claim: 1

$\overline{T}$ is well defined

Let $v_1+W = v_2+W$

$\Rightarrow v_1 - v_2 \in W$

$\Rightarrow (v_1-v_2)\,T \in W\,T$ $[\because\ W\,T \subseteq W]$

$\Rightarrow (v_1-v_2)\,T \in W$

$\Rightarrow v_1T - v_2T \in W$

$\Rightarrow v_1T + W = v_2T + W$

$\Rightarrow (v_1+W)\overline{T} = (v_2+W)\,\overline{T}$

Hence $\overline{T}$ is well defined

Claim: 2

$\bar{T}$ is a linear Transformation on V/W

Let $x,y \in V/W \Rightarrow x = v_1+W$

$y = v_2+W, \quad v_1,v_2 \in V$

Now $(x+y)\,\bar{T} = [(v_1+W) + (v_2+W)]\,\bar{T}$

$= [(v_1+v_2) + W]\,\bar{T}$

$= (v_1+v_2)\,T+W$ [by def of $\bar{T}$]

$= (v_1T + v_2T) + W$ (since T is linear)

$= (v_1T+W) + (v_2T+W)$ [by cocet +]

$= (v_1+W)\,\bar{T} + (v_2+W)\,\bar{T}$ [def of $\bar{T}$]

$= x\bar{T} + y\bar{T}$

Let $\lambda \in F$

$$(\lambda x)\bar{T} = (\lambda(v_1+W)\bar{T}$$

$$= (\lambda v_1+W)\bar{T} \text{ [scalar in V/W]}$$

$$= (\lambda v_1) T+W \text{ [definition of } \bar{T} \text{]}$$

$$= \lambda(v_1T) +W \text{ [} \because \text{ } T \in A(V)]$$

$$= \lambda(v_1T+W) \text{ [by scalar multiplication in V/W]}$$

$$= \lambda((v_1+W)\bar{T})$$

$$= \lambda (x\bar{T})$$

$\therefore$ $\bar{T}$ is a linear Transformation on V/W

Claim: 3 If T satisfies a poly f(x) $\in$ F[x] then so does $\bar{T}$

Let x = v+W $\in$ V/W

Now, $x\,(\overline{T^2}) = (v+W)\,(\overline{T^2})$

$= vT^2+W$

$= (vT)\,T+W$

$= (vT+W)\,\bar{T}$

$= (v+W)\,\bar{T}\,\bar{T}$

$= (v+W)\,(\bar{T})^2$

$= x(\bar{T})^2 \qquad \forall\, x \in V$

$\Rightarrow \boxed{\overline{T^2} = (\bar{T})^2}$

In general $\boxed{\overline{T^k} = (\bar{T})^k}$ for any non-negative integer k

Let $f(x) = a_0 + a_1 x + \text{-----} + a_m x^m$, $a_i \in F$

$$\Rightarrow f(T) = a_0 I + a_1 T + \text{-----} + a_m T^m$$

$$\Rightarrow \overline{f(T)} = a_0 I + a_1 \overline{T} + a_2(\overline{T})^2 + \text{-----} + a_m(\overline{T})^m$$

$$= f(\overline{T})$$

Suppose T is satisfies f(x) $\Rightarrow f(T) = 0$

$$\Rightarrow \overline{f(T)} = 0$$

$$\Rightarrow f(\overline{T}) = 0$$

$\Rightarrow \overline{T}$ satisfies f(x)

Claim: 4 $P_1(x)$ divides p(x)

Let $p_1(x)$ be the minimal polynomial of $\overline{T}$ over F

Let P (x) be the minimal polynomial of T over F

Given: p(x) is the minimal polynomial of T

$\Rightarrow$ p(T) = 0 & q(T) $\neq$ 0 such that deg (q(x)) < deg (p(x))

$\Rightarrow$ p(T) = 0 [$\because$ T satisfies p(x) $\Rightarrow \overline{T}$ satisfies p(x)]

$\Rightarrow p_1(x) \mid p(x)$ [$\because$ $p_1(x)$ is the minimal polynomial of $\overline{T}$ & p(x) is a polynomial satisfied by $\overline{T}$]

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Theorem:

If $T \in A(V)$ has all its eigen values in F then there is basis of V in which the matrix of T is triangular

Proof:

We prove this theorem by induction on $\dim_F (V)$

If dim (V) = 1 then dim (A(V)) = 1 so every element of A(V) is a scalar

The theorem is true for this case.

Now, we assume that the theorem is true for all vector space over F of dimension n-1 & let V be of dimension n over F.

It is given that $T \in A(V)$ has all its eigenvalues in F

Let $\lambda_1 \in F$ be an eigen value of T

$\Rightarrow$ there exists $v_1 \neq 0$ in V such that $v_1 T = \lambda_1 V_1$

Let $W = \{av_1 / a \in F\}$

$\Rightarrow$ W is the subspace whose basis is $\{v_1\}$

$\Rightarrow$ dim (W) = 1

<u>Claim:</u> W is invariant under T ie, $WT \subseteq W$

Suppose $x \in W \Rightarrow x = av_1$, $a \in F$

Now, $xT \in W\,T$ & $xT = (av_1)\,T$

$= a(v_1 T)$

$= a(\lambda_1 v_1)$

$= (a\lambda_1)\, v_1$

$= bv_1 \in w$

$\therefore\ xT \in W\,T \Rightarrow x\,T \in W$

$\Rightarrow WT \subseteq W \Rightarrow$ W is invariant under T

Def $\bar{V}$ = V/W

$\Rightarrow$ dim ($\bar{V}$) = dim V – dim W = n-1

Then, T induces a linear transformation $\bar{T}$ on $\bar{V}$

such that, $p_1(x)$ divides $p(x)$ [$\because$ by lemma]

where, $p_1(x)$ is the minimal polynomial of $\bar{T}$ over F &

$p(x)$ is the minimal polynomial of T over F

$\Rightarrow$ every root of $p_1(x)$ is also a root of $p(x)$

$\Rightarrow$ $\bar{T}$ has all its eigen values in F

since T has all its eigen values in F

Now dim ($\bar{V}$) = n-1 & $\bar{T} : \bar{V} \rightarrow \bar{V}$ has all its eigen values in F

So, by induction hypothesis, there exists a basis {$\bar{v}_2, \ldots, \bar{v}_n$} of $\bar{V}$

such that the matrix of $\bar{T}$ is triangular.

$$\begin{array}{c} \\ \bar{v}_2 \\ \bar{v}_3 \\ \vdots \\ \Rightarrow \bar{v}_i \\ \bar{v}_n \end{array}
\begin{array}{c} \begin{array}{cccccc} 2 & 3 & \ldots & i & \ldots & n \end{array} \\
\begin{bmatrix} a_{22} & 0 & \ldots & 0 & \ldots & 0 \\ a_{32} & a_{33} & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ a_{i2} & a_{i3} & \ldots & a_{ii} & \ldots & 0 \\ a_{n2} & a_{n3} & \ldots & a_{ni} & \ldots & a_{nn} \end{bmatrix} \end{array}$$

$$\Rightarrow \left.\begin{aligned} \bar{v}_2\,\bar{T} &= a_{22}\,\bar{v}_2 \\ \bar{v}_3\,\bar{T} &= a_{32}\,\bar{v}_2 + a_{33}\,\bar{v}_3 \\ &\vdots \\ \bar{v}_i\,\bar{T} &= a_{i2}\,\bar{v}_2 + a_{i3}\,\bar{v}_3 + \ldots + a_{ii}\,\bar{v}_i \\ &\vdots \\ \bar{v}_n\,\bar{T} &= a_{n2}\,\bar{v}_2 + a_{n3}\,\bar{v}_3 + \ldots + a_{nn}\,\bar{v}_n \end{aligned}\right\} \text{———— } (1)$$

Now,

$$\bar{v}_2\,\bar{T} = a_{22}\,\bar{v}_2$$

$$\Rightarrow (v_2+w)\,\bar{T} = a_{22}\,(v_2+w)$$

$$\Rightarrow v_2T + w = a_{22}\, v_2 + W$$

$$v_2T - a_{22}\, v_2 \in W = \{av_1 / a \in F\}$$

$$\Rightarrow v_2\, T - a_{22}\, v_2 = a_{21}\, v_1$$

$$\Rightarrow \boxed{v_2\, T = a_{21}\, v_1 + a_{22}\, v_2}$$

Similarly $\bar{v}_i\ \bar{T} = a_{i2}\ \bar{v}_2 + a_{i3}\ \bar{v}_3$ +-----+ $a_{ii}\ \bar{v}_i$

$$\Rightarrow \boxed{v_i\, T = a_{i1}\, v_1 + a_{i2}\, v_2 + \text{------} + a_{ii}\, v_i}$$

Thus including $v_1 T = \lambda_1 v_1$ we have obtained

$$v_1 T = a_{11} v_1 \text{ where } a_{11} = \lambda_1,$$

$$v_2 T = a_{21} v_1 + a_{22} v_2$$

$$\vdots$$

$$v_i T = a_{i1} v_1 + a_{i2} v_2 + \text{-----} + a_{ii} v_i$$

$$\vdots$$

$$v_n T = a_{n1} v_1 + a_{n2} v_2 + \text{----} + a_{nn} v_n$$

By the definition of the matrix of a linear transformation we see that the matrix of T is

$$\begin{pmatrix} \lambda_1 & 0 & 0 \ldots 0 \\ a_{21} & a_{22} & 0 \ldots 0 \\ \vdots & & \\ a_{n1} & a_{n2} & a_{n3} \quad a_{nn} \end{pmatrix}$$

which is triangular

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